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The Relation Between Income Growth and Unemployment

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On an optimistic-realistic view, the best hope [for the 1970's] is that a 4 percent rate of unemployment and a 2 percent rate of price increase will prove compatible and that such a combination will be regarded as a satisfactory compromise by the American public. This was the hope before the Vietnam spurt in mid-1965, and nothing that has happened since demonstrates that it is unattainable.[1]*

These words, dating from late 1969, are those of Arthur Okun, one of the principal architects of the Federal government's economic policy in the 1960's. His "best hope" may be taken as the consensus view in 1969 among economists of what the 1970's might bring. And in fact, neither number was wildly out of line with postwar experience up to that point; from 1947 through 1969, the unemployment rate averaged 4.6 percent and inflation proceeded at an average rate of 2.6 percent.

Nor was there good reason on the surface to believe that the targets of 4-percent unemployment and 2-percent inflation were incompatible. The two serious inflations of the postwar period—those associated with the Korean and the Vietnam wars—were accompanied by unemployment well below 4 percent, at a 3.1-percent average in 1951-53 and at a 3.6-percent average in 1967-69. In addition, there were two peacetime periods when policy brought unemployment near 4 percent—1955-57 with average unemployment of 4.3 percent and 1965 with an average of 4.5 percent—and both were associated with inflation near 2 percent. It is now clear, however, that either something was wrong with Okun's analysis or that the economy has changed dramatically in recent years: from 1970 through 1973, unemployment averaged 5.3 percent, and inflation averaged 4.7 percent.

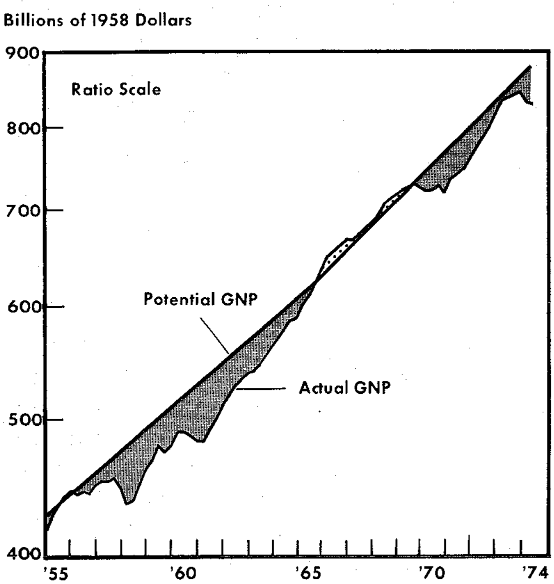
Underlying this discussion is the relation known as Okun's Law, which in its most general form states that there is a highly stable, predictable relation between the unemployment rate and the rate of growth of real income. Okun's development of this concept led to his construction of two of the best-known tools of fiscal and monetary analysis—potential GNP and the high-employment Federal budget. The measures, in the form which came into common use in the mid- and late-1960's, measure GNP and the Federal deficit, not as they are, but as they would be if the unemployment rate were at 4 percent.

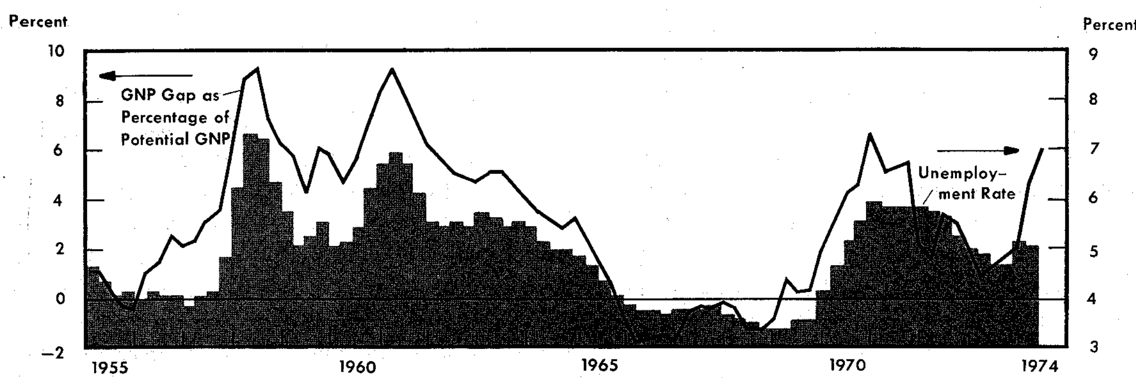
Okun's estimate was that on the basis of the historical growth in the labor force, capital stock and productivity, an annual rate of growth of real income of approximately 4 percent would produce a constant unemployment rate.[2] He also estimated that each 1.0 percent of growth above the 4-percent figure would, on a quarterly basis, reduce the unemployment rate by .08 percent. Thus, 5 percent real growth, sustained for a full year, would lower

*Footnotes at end of article.

unemployment by 0.3 percent. With these two numbers, it is easy to compute the added real income at each point in time which is needed to produce 4 percent of unemployment. This added income, which has also been used widely in fiscal and monetary analysis, is called the GNP gap. The high quality of the relation between the gap and unemployment is apparent from the chart, which presents the observed relation between potential and actual output and unemployment (Chart 1).

Chart 1





Natural rate of unemployment

The bottom panel in the chart makes it clear that the computation of potential GNP based on 4-percent unemployment is arbitrary. The quality of the fit to observed unemployment would be unchanged if the line for zero gap were shifted up or down to a different level of full employment. This is so because the GNP gap approach essentially reflects only supply conditions: an added unit of labor in this world always generates and requires added growth in real income. There is no connection with the related demand notion that high demand for labor should also push wages up relative to other prices and thus automatically lower that demand. That this world cannot always be so may be seen by taking the case—which occurred in both the Korean and Vietnam wars—where neither added labor nor added capital were to be had. Any attempt to increase output must fail; if aggregate monetary or fiscal policy were used to this end, the effect would simply to be increase prices.

The necessary addition to the discussion is provided by Milton Friedman's concept of the natural rate of unemployment. Friedman observes that the unemployment rate is basically a measure of the degree of tightness in labor markets, and hence the amount of pressure on wage rates. At each point in time there will be one unemployment rate—the natural rate—which involves *no change* in the amount of wage pressure on the rate of price inflation. The rate of increase in wages at the natural rate will just equal the rate of growth of productivity plus the rate of inflation expected by participants in the labor market. Gearing policy to any unemployment rate lower than the natural rate will cause problems, for the resultant wage pressure on prices will eventually mean that the inflation rate will go up, and will require a further wage rate increase to offset the new inflation rate. Thus the inflation rate could increase without limit, unless the authorities decide at some point to pursue policies which raise unemployment to at least the natural rate.

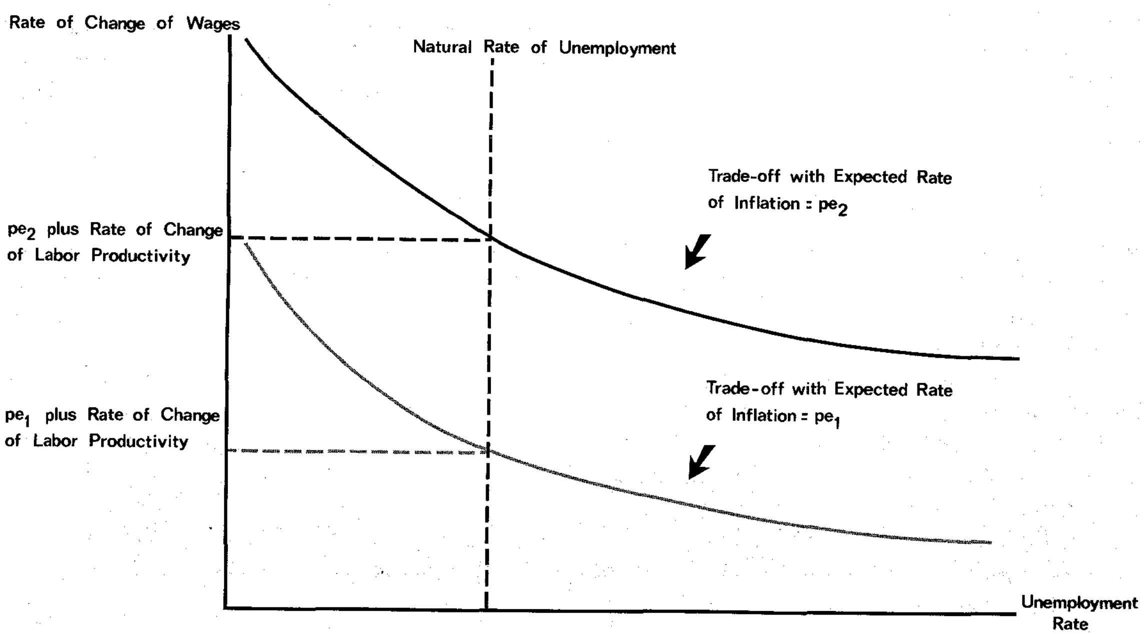
This point is illustrated by Chart 2, which depicts the familiar Phillips curve trade-off between the growth in wages and the unemployment rate. The natural-rate analysis permits us to say that changes in the expected rate of inflation shift the short-run trade-off up or down. In the chart, an increase in the rate of expected inflation from pe_1 to pe_2 increases the rate of increase of wages associated with any unemployment rate by the same amount—the natural unemployment rate, associated with no change in the amount of wage pressure on the inflation rate.

The main requirement for the existence of a natural rate is that changes in prices operate independently of the real economy; that is, that an increase in wage pressure on prices will eventually be fully reflected in the price level but will have no effect on the level of real income. This assertion is natural in economics because of the usual assumption that people are not subject to "money illusion": at least in the long run, workers and businesses respond to shifts in real (and not in money) wages and assess their welfare accordingly. Thus Friedman's approach would suggest that potential income should *only* be computed with reference to the natural rate. Use of any other rate would imply that the calculation is not neutral to inflation, that it embodies some amount of misallocation due to changes in inflation.

It is the purpose of this article to measure the natural rate of unemployment implicit in Okun's Law and examine the relation for changes over time. It will be shown that, except for the effect of temporary conditions:

1. The natural rate of unemployment has remained unchanged at 4.8 percent during the post-war period;
2. The growth rate of potential income has increased from 3.5 percent before 1956 to 4 percent since 1965; and

Chart 2



3. The relation between income growth and unemployment has changed in a way which makes it much less likely that high income growth will produce very low unemployment rates (below 4 percent) than was the case in the early post-war period.

Okun's Law and the natural rate

There is no reason in principle for either Okun's Law or the natural rate of unemployment to be particularly stable over time. Both depend on a number of factors which can shift over time. For example, with respect to Okun's Law relating the growth in potential GNP to the growth of the input of capital and labor, the growth rate of the civilian labor force is far from constant, and is influenced by such items as the number of young persons entering the labor force and the number of women entering or reentering the labor force, both matters largely beyond the scope of economics. But it may well be that all such influences are transitory, and that a simple correction for them will produce relations which are indeed very stable. Such is the case for Okun's Law, at least for the period from the end of World War II through the mid-60's, the period studied by Okun and his followers. There is also some evidence, provided by Friedman in the form of an examination of the relation between inflation and unemployment, that the natural rate of unemployment was also largely unchanged in the same period.

Whether this stability remains in the 1970's is a question of some importance for policy making, because Okun's Law and the natural rate both bear directly on the major social concerns of policy—the high social cost of unemployment and the misallocation of resources implicit in inflation. We will examine this stability with the aid of a minor extension of the basic Okun model to allow for a measure of the natural rate of unemployment. The Okun model makes the change in the unemployment rate ΔU_t at time t equal some number a (the *income multiplier*) times the difference between the rate of growth of real income YR_t and its potential rate of growth g:[3]

$$(1) \qquad \Delta U_t = -a(YR_t - g)$$

If we sum all quarterly changes in unemployment from 1 to time t, and add the level of the unemployment rate U_0 at time zero, we obtain the *level* of the unemployment rate U_t at time t as a function of the *sum* YS_t over all past rates of income growth and a time trend. This last term arises because a sum over a constant growth rate g will increase by just one unit of growth per quarter:

$$(2) \qquad U_t = U_0 - a(YS_t - gt)$$

Two other items must be taken into account before fitting the equation. First, there is no logic that requires all of the effects of a certain level of income growth to occur in the present quarter. Our investigation of this point showed that the best statistical results would be obtained if we used instead a simple average of the current and just-past values for the change in real income. We replace the YS_t term in equation (2) with:

$$YA_t = (YS_t + YS_{t-1})/2$$

Secondly, the relation will be subject to a variety of temporary elements, and there must be a correction for such disturbances in the relation if our estimates are to be accurate. A simple correction is to allow the unemployment rate to depend on the observed error e_{t-1} in the relation last quarter times a correction factor p_p as well as on the other, more basic, variables. What this dependence means is that if the basic projection was high last quarter, it will be high again this quarter because the transitory forces which caused the high projection will not yet be completely spent. These additions yield the final form of the relation:

$$U_t = U_0 - a(YA_t - gt) + pe_{t-1}$$

U_t is the overall unemployment rate
U_0 is the equilibrium unemployment rate in 1948

a is the income multiplier

YA_t is a two-quarter moving average of the sum over all annual rates of growth since 1948

g is the growth rate of potential income

pe_{t-1} is a correction factor times the error in the relation in the previous quarter.

With these alterations, deviations in the relation either show up as random events of no importance or are subsumed in the natural growth rate of income. In other words, there can be no trend in the natural rate of unemployment. Such is not the case for the equation as originally written in equation (1), for it would not be possible to distinguish between differences caused by changes in the rate of income growth and by changes in the natural rate of unemployment. Both would appear as changes in the value of the constant. Either type would, of course, be a true structural change in the economy, but they each would have very different meanings.

The natural rate is a reference point for the amount of wage pressure exerted on the economy by a given unemployment rate. If the rate does not change over time, the essential meaning of a given unemployment rate does not change. The potential growth rate for income and the income multiplier are in contrast summary measures of the relations among productivity, labor force growth, and employment. If they do not change, the nature of technological and labor-force growth underlying Okun's Law does not change. All of these measures are subject to some extent to the offsetting forces of substitution, but especially so the natural rate of unemployment. There is good reason to believe that a relative increase in, say, the numbers of some low-skill, high-unemployment group of the labor force would be partly offset by a lowering of that group's wage rate relative to others. There is no similar automatic presumption that an increase in technology would be offset by a decrease in labor force growth, or that it would leave unchanged the potential growth of income.

It is worth noting in this regard that there is no presumption in the present tests as to the relative importance of labor and other factors in generating inflation. It may be that the source of the stability in the natural rate we will observe is a systematic tightening in the supply of capital or of production inputs to the United States by the rest of the world. Should such be the case, there would be no change in the implications of our analysis for *aggregate* policy aimed jointly at all sectors, but there would be a possibility of devising *specific* policies aimed at the relief of particular bottleneck sectors and thus perhaps at a permanent lowering of the natural rate of unemployment. Thus, the results we obtain should be read entirely as having implications for aggregate policy.

Interpretation of results

We fit the adjusted relation to the unemployment rate for the period 1948.3 through 1974.1—the longest period available with quarterly data for all series—and obtained the following results. Standard errors for the estimated coefficients are shown in parentheses below the equation; income growth is at annual rates.

$$U_t = \underset{(.35)}{5.82} - \underset{(.005)}{.092}(YA_t - \underset{(.21)}{3.76}t) + \underset{(.051)}{.856}e_{t-1}$$

The standard error in fitting this quarterly average unemployment rate is .217 percent, which compares well with the monthly sampling error of .20 percent for the rate published by the Bureau of Labor Statistics.

This relation may be interpreted as follows:

1. If the two-quarter average growth of real income is above a 3.76 percent annual rate, and no temporary factors are at work, the unemployment rate will fall. Below that growth rate, unemployment will rise.

2. If the two-quarter average growth rate is one percent above this critical value and no temporary factors are present, the unemployment rate will decline by .092 percent.

3. If some temporary factor is present in quarter t, then 85.6 percent of the fac-

tor will be present in quarter (t+1).

4. The value of the constant, 5.82 percent, is the sum of the natural rate of employment, defined entirely in terms of productivity and labor force factors,[4] and the amount of displacement of the unemployment rate from that level in 1948. In calculating the natural rate, we would find that average unemployment in the period of fit was 4.80 percent, and that the average growth rate of real income was 3.77 percent. Because the latter is above the natural growth of income, the natural rate of unemployment must be above the observed average rate. Though the difference in growth rates is small, over the 102 quarters of the sample period, it is not trivial, and the necessary correction comes to .04 percent, making the natural rate of unemployment 4.84 percent.

These results largely validate Okun's results: the 3.8 percent potential growth rate for income and .09 income multiplier are not appreciably different from Okun's 4.0 and .08 figures. It does appear that the natural rate of unemployment is now, and has been for the entire postwar period, far above 4 percent.

We can check on this result by asking whether the underlying relation between income and employment has changed over the years. The answer to this question is a qualified yes: we can be 95 percent certain that the relation has changed, but we cannot be 99 percent certain.[5] Despite this mild uncertainty, we get large policy implications from the observed changes, because they substantially affect our ability to achieve very low rates of unemployment. This uncertainty does mean that the forecasting error of the relation gets improved by only a very small amount by allowing for a shift. The expected error in our relation is .217 percent in the equation above; after the shift, the expected error drops to .208 percent. Such an improvement in fit would not be worth the added complexity of the relation were it not for its policy implications.

The allowance for shift was made by dividing the period of fit into three equal parts, ending respectively in 1956.4, 1965.2, and 1974.1. The estimated coefficients were set to be constant in the first and last periods, with a smooth transition from the initial to the final values occurring during the second period in order to provide continuous values for all of the estimates throughout the period. The differences which arose were then checked for statistical validity, and only the constant term and the income multiplier proved to have shifted; the multiplier on the time trend did not change. This result is important, for the role of the parameter on the time trend is to determine the relation between the natural rate of growth of income and the natural rate of unemployment: if it does not change, the natural rate of unemployment does not change, not matter what happens to the natural growth of income, because this parameter is the product of the income multiplier and the growth rate of potential income. The same calculation used above yields a natural rate of unemployment of 4.82 percent, essentially the same number obtained before. Thus, the natural rate is now, and has been for the entire postwar period, very close to this number.[6]

However, two important shifts have occurred: an increase in the natural rate of growth of income and a decrease in the multiplier on income. The two equations appear below; the numbers have the same interpretation as before.

1. Fully effective 1948.3-1956.4; declining in importance 1957.1-1965.2.

$$U_t = \underset{(.45)}{6.66} - \underset{(.005)}{.099}\,(YA_t - \underset{(.21)}{3.51}) + \underset{(.055)}{.831}e_{t-1}$$

2. Increasing in importance 1957.1-1965.2; fully effective 1965.3-1974.1.

$$U_t = \underset{(.95)}{4.03} - \underset{(.005)}{.087}\,(YA_t - \underset{(.24)}{4.01}t) + \underset{(.055)}{.831}e_{t-1}$$

Thus, achieving a decline in the unemployment rate now requires much higher income growth (4.01 percent against 3.51 percent) than it did earlier, while the response of unemployment to given income growth is lower now than before (.087 against .099). This asymetric change means that in boom periods, the higher natural rate of income growth and the lower income multiplier both work to keep unemployment from going as low as it would have under the earlier relation, while in recessions the two largely offset each other, with the high natural growth rate still working to keep unemployment up but with the low multiplier now working to keep it down. This point is made in concise form in Chart 3, which contains projections made with two relations for the period from 1970.1 to 1974.1—a period which contains both a recession (from 1970.1 to 1971.1) and a rather long period of very high income growth (from 1971.4 to 1973.2). These projections are made by ignoring the allowance for transitory components (the multiplier on e_{t-1}) and thus track only the basic Okun's Law relation. In this chart, the two relations have almost identical performance in the recession period (the first five quarters), but the relation based on the earlier data begins to deteriorate sharply in the quality of its projection once income growth moves to high levels in 1971.4. The deterioration continues throughout the high growth period, and only begins to unwind in the most recent three low-growth quarters. The relation in fact projects that the rates reached near the 1973 minimum would be well below the lowest peacetime unemployment rates of the postwar period, reached in the 1955-56 expansion.

The deterioration would not be extremely serious, and there would be no significant differences between the beginning and end of the period, if the deterioration were regarded as arising from a transitory component (Chart 4). But this correction will not work. The two relations once again show almost the same performance during the 1970 recession, with the quality of fit being much better than in Chart 3. After the recession, however, the relation based on the recent period continues to track the actual unemployment rate rather closely, while the relation based on older data steadily underprojects the actual numbers.

Conclusions of study

1. For the postwar period as a whole, the best values for the basic formulation are 3.8 percent for the natural rate of growth of income and .09 for the quarterly multiplier between real income and the unemployment rate. These values agree rather well with the

Chart 3

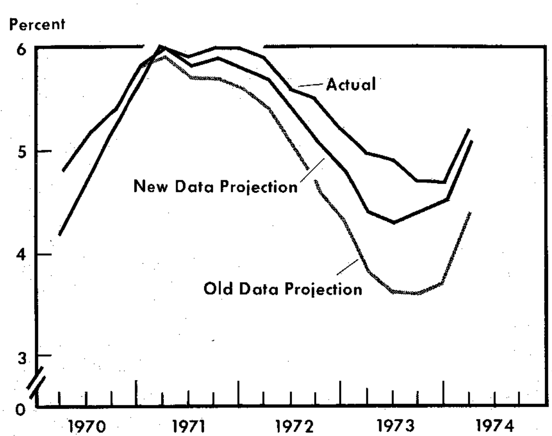
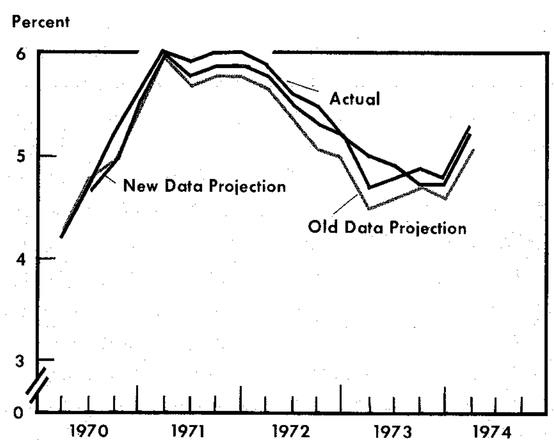


Chart 4



principal Okun results of a 4.0 percent natural growth rate and a .08 income multiplier. Thus the events of the last seven years have not invalidated the main thrust of Okun's Law.

2. The principal applications of the Law—the constructions of potential GNP, the GNP gap, and the full-employment budget—have all been based on a 4.0 percent unemployment rate, and ignore the fact of a 4.8 percent natural rate of unemployment. This natural rate has not changed at all during the postwar period, and thus stands as at least as durable an artifact as Okun's Law itself. It follows that policy based on a 4.0 percent full-employment rate will be an engine of inflation. Policy, both now and in the past, either should have aimed at an unemployment target of 4.8 percent—not 4.0 percent—or it should have built much more comprehensive inflation protection for individuals and businesses into the economy's institutions than it has.

3. Though the added data for the 1970's generally support Okun's number for the postwar period as a whole—with his based on data through the mid-1960's—they also suggest that there has been a change from the relation which held in the 1950's. The shifts involved are modest: the natural growth of real income has increased from 3.5 to 4.0 percent, and the multiplier from income growth to the unemployment rate has declined from .10 to .09, with no change in the 4.8 percent natural rate of unemployment. However, these small shifts have an important policy implication. When income growth is low, the two changes act to offset each other, and the recession behavior of Okun's Law has thus remained largely unchanged over the years. But when income growth is high, the two shifts reinforce each other and prevent the unemployment rate from going as low as it formerly did. The basic relation projected a minimum unemployment rate of 3.6 percent for the 1971-73 period, while the recent relation projected a minimum of 4.3 percent. It is thus no longer possible to get to 4.0-percent unemployment during even the strongest of peacetime booms.

4. Because the natural rate of unemployment has not changed, it is not possible to describe the shifts which have occurred as being in any meaningful sense "structural." A structurally unemployed person gets that way because he or she possesses a mix of skills which, because of technological factors and the skill mix of the labor force, does not lend itself to finding a job at the going wage. These elements are precisely those which determine the natural rate of unemployment, so that it is hard to conceive of a true structural shift which does not alter that rate. Undoubtedly, there has developed over time a different composition of the labor force and a different mix of available jobs, but the evidence suggests that these changes have been offset precisely by substitution between labor-force categories and by shifts into newly available jobs. This argument does not necessarily mean that there is no problem of structural unemployment: it means instead that the problem has not gotten any worse (or better) over the years.

Footnotes

1. Okun, *The Political Economy of Prosperity* (Washington: Brookings, 1970) p. 102.
2. This statement is a simplification of the subtle Okun approach to the measurement of potential output. After careful analysis, Okun concluded that potential output—adjusted to 4-percent unemployment—grew at an average of 4½ percent in the late 1940's through 1953, at 3½ percent through 1961, and at 4 percent since that time. Our results will largely—though not completely because of different sample periods—validate those numbers. *Ibid.*, ch. 2.
3. This equation is the inverted form of Okun's potential-output equation, written in a way which begs the question of what an appropriate growth rate for output is. The first close analog of this equation appeared in the Council of Economic Advisers' statement to the Joint Economic Committee of March 1961.
4. The Friedman definition takes into account the pattern of individual adjustments to inflation, and thus cannot be used without a formal inflation-generating mechanism. Because the subject here is Okun's Law, we use a narrower definition, without reference to either the prevailing rate of inflation or the pattern of adjustment to that rate. In practice, Friedman's natural rate should be essentially identical with ours. For a formal model of the nominal price apparatus, see Milton Friedman, "A Theoretical Framework for Monetary Analysis," *Journal of Political Economy* (1970), pp. 193-238. To our knowledge, the earliest exposition of the natural-rate concept is in Friedman's "The Role of Monetary Policy," *American Economic Review* (1968), pp. 1-17.
5. The statistical results are in Appendix A.
6. A mathematical treatment of this observation is given in Appendix B.

Technical notes

A. Statistical method. The statistical tests employed above use Scheffe's S-method for the analysis of variance. This method requires that one establish a "pool" of acceptable independent variables for a relation, none of which depends for the desirability of its inclusion in the relation on the presence or absence of any other variable in the relation. The method then operates by casting variable out of the pool: if there are k variables, the method operates by casting out the least important variable, the two jointly least important variables, and so on. The results are normally presented in reverse order of casting out, and thus have the same appearance as a stepwise regression. The difference is that the size of the pool determines the number of lost degrees of freedom, and the method is thus relatively invulnerable to attempts at "data-mining."

The attraction of analysis of variance as an econometric technique has long been obvious, but its use has been severely limited because an appropriate technique for setting up pools for distributed lag models is not at all obvious. A second problem, the stochastic time dependence of most economic time series—which renders invalid the only non-robust aspect of analysis of variance, its assumption of time independence—has been solved by the advent of the Cochrane-Orcutt and Hildreth-Liu serial correlation corrections. The correction used in this paper is Cochrane-Orcutt because it is a true non-linear least-squares technique and thus fits explicitly into the analysis-of-variance framework. The "pool" problem for distributed lags is that we normally have no very clear notion of how long a distributed lag should be, and we often have only the vaguest idea of the proper shape for such a lag: the only more-or-less universal requirement is that the structure be continuous. The standard solution to distributed-lag estimation is the polynomial-distributed lag, a device which does not lend itself to "pooling" very well: there are 136 different polynomial lags of length between one quarter and sixteen, a "pool" which would more than exhaust the number of postwar quarterly observations on the economy. Our solution to this problem was to select one polynomial of each desired lag length, each set to form a continuous curve through to the maximum lag and have one estimating parameter left over. The simplest such family of curves is the set of parabolas forced to zero at either end of the lag structure. A typical member of the "pool" for variable X_t would be:

$$Z_t = \left(\sum_{i=0}^{k} (i+1)\,(k-1)\,X_{t-i}\right) \Bigg/ \left(\sum_{i=0}^{k} (i+1)\,(k-i)\right)$$

The pool variable for the income growth rate YR_t above for $k=2$ is exactly the paper's YA_t variable. Once YA_t entered the relation, no other member of a pool having k go from 1 to 8 had more than the slightest effect on unemployment.

Having checked for lag structure, we then checked for structural change. These tests were separated only because we wanted to use the largest possible sample in the structural change test; with the short structure found, it was necessary to lose only one observation to the distributed lag. The main relevant information from the analysis appears in the table below.

Variables in regression	*Sum of squared residuals*	$F_1{}^*{}_{,n-k}$
1. Constant, income, time trend	19.2200	—
2. (1) plus autoregressive correction	4.6983	305.99_1
3. (2) plus structural split on income	4.5214	3.83_3
4. (3) plus structural split on constant	4.2418	6.40_2
5. (4) plus structural split on time trend	4.2275	$.33_x$

*F's are placed on line for which the variables included are the maintained hypothesis.

1: Significant with at least 99.9-percent confidence.
2: Significant with 97.5-percent confidence, but not with 99 percent.
3: Significant with 90-percent confidence, but not with 95 percent.
x: Not significant with 90-percent confidence.

An easy summary device for this analysis is the triangle of F's, which appears below. The $F_{1,n-k}$'s appear on the diagonal.

Test hypothesis / Maintained hypothesis	(1)	(2)	(3)	(4)
(2) (p correction)	305.99_1			
(3) (income split)	159.29_1	3.83_5		
(4) (constant split)	114.17_1	5.22_2	6.40_3	
(5) (trend split)	85.11_1	3.56_3	3.38_4	$.33_x$

1: Significant with at least 99.9 percent confidence.
2: Significant with 99 percent confidence, but not with 99.9 percent.
3: Significant with 97.5 percent confidence, but not with 99 percent.
4: Significant with 95 percent confidence, but not with 97.5 percent.
5: Significant with 90 percent confidence, but not with 95 percent.
x: Not significant with 90 percent confidence.

The only variable not worth having in the relation is the structural split on the trend line, which with its very low .33 $F_{1,n-k}$ ratio does not represent a reduction in variance over the relation excluding the variable. We may be 97.5 percent certain that the constant does have a structural split; as a maintained hypothesis, it represents a reduction in variance against all test hypothesis with at least that confidence. We have the same confidence that the structural split on income is significant. Though its $F_{1,n-k}$ as a maintained hypothesis is low (significant only at 90 percent), its $F_{2,n-k}$ with the constant split also included, at a value of 5.22, is significant at 99 percent. We then reduce this confidence so that we have in the constant split because that split is necessary to our confidence in the income shift. And, finally, the Cochrane-Orcutt correction is necessary to the relation to a very high degree of confidence.

B. Computing the natural rate of unemployment. The relations above may be rewritten as errors-in-variables relations with the unemployment role as the sum of the natural rate of unemployment and the implicit displacement of unemployment from this rate at time t:

$$U_t = (U_{xt} + U_n) - a(YA_t - gt)$$

For the natural rate taken as a constant, as may be done by the obvious errors-in-variables interpretation of U_{xt}, we may compute the natural rate for any period for which the sum over the displacements is zero. This sum is zero for the full sample by construction of the results of running least-squares fits of the type presented in this paper. If we take the mean of the above relation, we must have:

$$\overline{U} = U_n - a(\overline{YA} - \bar{g}\bar{t})$$

We may ignore the moving-average aspect of YA_t because each observed growth rate enters the sum which forms YA_t with unit weight (aside from a minor endpoint problem) and write:

$$\overline{YA} = \frac{1}{n}\left(\sum_{t=1}^{n}\left(\sum_{i=1}^{t}(YR_i - g)\right)\right)$$

$$= \frac{1}{n}\sum_{t=1}^{n}(\overline{YR}_t - g)t$$

An easy and highly accurate approximation to this sum for any long series which contains much movement (as is true of income growth) is to let YR_t equal YR_n, the mean for the full sample. We get:

$$\overline{YA} = \frac{1}{n}\sum_{t=1}^{n}\left(\overline{YR}_n - g\right)t$$

$$= \frac{1}{n}\left[YR_n - g\right]\frac{n(n+1)}{2} = \frac{n+1}{2}\left(\overline{YR}_n - g\right)$$

If we apply this formula to the above, we get

$$U_n = \overline{U} + a\,\frac{n+1}{2}\,(\overline{YR}_n - g)$$

For the full 1948.3-1974.1 series (103 observations) the relevant calculation is:

$$U_n = 4.80 + .092 \times 52 \times (3.774 - 3.765) = 4.84$$

Note that we must carry one extra digit of accuracy in the income growth rates to perform the calculation to the same accuracy as the other numbers.

The structural split does not complicate the calculation of the natural rate overmuch; it remains true that the sum over the displacements involved is still zero for the full sample, and the natural rate for the full sample comes to 4.82 percent. An indication of how this is derived may be obtained by splitting the relation into two pieces, one relevant to the early set of parameters and one relevant to the late set of parameters. In matrix form, and ignoring the transitory component correction, the relation is:

$$\begin{bmatrix} U_{48.3} \\ \vdots \\ U_{56.4} \\ U_{57.1} \\ \vdots \\ U_{65.2} \\ U_{65.3} \\ \vdots \\ U_{74.1} \end{bmatrix} = \left[\begin{bmatrix} 1 \\ \vdots \\ 1 \\ 33/34 \\ \vdots \\ 1/34 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} YA_{483} \\ \vdots \\ YA_{56.4} \\ YA_{57.1} \times 33/34 \\ \vdots \\ YA_{65.2} \times 1/34 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 33 \\ 34 \times 33/34 \\ \vdots \\ 67 \times 33/34 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right] \begin{bmatrix} U_{01} \\ -a_2 \\ a_1 g_1 \end{bmatrix} + \left[\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1/34 \\ \vdots \\ 33/34 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ YA_{57.1} \times 1/34 \\ \vdots \\ YA_{x\,65.2} \times 33/34 \\ YA_{65.3} \\ \vdots \\ YA_{74.1} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 34 \times 1/34 \\ \vdots \\ 67 \times 33/34 \\ 68 \\ \vdots \\ 103 \end{bmatrix} \right] \begin{bmatrix} U_{02} \\ -a_2 \\ a_2 g_2 \end{bmatrix}$$

We may now calculate the natural rate for the two pieces of this relation on the assumption that the net displacement of the unemployment rate in each half is zero. This assumption will not quite be true: some covariance will arise in the middle of the period. This covariance should be quite limited, however, because of the declining weights on the early parameters in the middle third and the rising weights on the late parameters. When we perform the calculations, we get:

1. Fully effective 1948.3-1956.4; declining in importance 1957.1-1965.2.

$$U_n = 4.71 + .099 \times 26 \times (3.567 - 3.513) = 4.85$$

2. Increasing in importance 1957.1-1965.2; fully effective 1965.3-1974.1.

$$U_n = 4.89 + .087 \times 25.5 \times (3.972 - 4.013) = 4.80$$

The growth rate for income obtained in the first calculation is above the mean of 3.30 percent obtained by simply splitting the sample in two at 1961.2 because of the effect of high income growth in the boom of 1962-65. The figure in the second calculation gets reduced from the second half mean of 4.18 percent because of the 1957 and 1960 recessions. The figures are evidently quite close together, much closer than the disparity in the mean unemployment rates would suggest.